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47-835 Graph Theory
Homework 1 **Due:** Monday, Sept 13, 2006

1. **Problem 1.1.30** (page 17) Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry in A^2 and in MM^T. What do the entries in position (i,j) of A^2 and MM^T say about G?

   Assume the vertices in G are indexed from 1 to n. The ith diagonal entry in A^2, call it $d(i)$, is computed as follows: $d(i) = A(i,1)A(1,i) + A(i,2)A(2,i) + ... + A(i,n)A(n,i)$. Since the adjacency matrix of any simple graph is symmetric and contains only 1's and 0's, this is equivalent to $d(i) = A(i,1)^2 + A(i,2)^2 + ... + A(i,n)^2$ where each term $t = 1...n$ on the right hand side evaluates to either a 1 or a 0 depending on whether there is an edge from v_i to v_t. Since there is a term for each node, all possible edges are accounted for, so $d(i)$ evaluates to the total number of edges incident on v_i.

   The ith diagonal entry in MM^T, call it $d_M(i)$ is the degree of v_i by similar reasoning. Assume the edges in G are indexed from 1 to m. We know$d_M(i) = M(i,1)M(i,1) + M(i,2)M(i,2) + ... + M(i,m)M(i,m)$. Since the incidence matrix of a simple graph has only 1's 0's, these terms $t = 1...m$ evaluate to 1 if edge t is incident on v_i and 0 otherwise. ∎

   The entry in position $(i.j)$ of A^2 indicates the number of paths of length two that connect v_i to v_j. This can be seen again by expanding the calculation of $A^2(i,j)$ and examining the meaning of each term.

   The entry in position (i,j) of MM^T indicates the number of edges between v_i and v_j. This is seen by examing each term in the calculation of $MM^T(i.j)$ and noting that it evaluates to 1 if the edge t is incident on both node i and node j. Since G is simple, these entries will always be 1 or 0.

2. **Problem 1.1.38** (page 18) Let G be a simple graph in which every vertex has degree 3. Prove that G decomposes into claws if and only if G is bipartite.

   Proof of $\Longrightarrow$ (if G decomposes into claws then it is bipartite): For any given claw c in the decomposition of G, there is a root node and three leaf nodes. No leaf of c can be the root of any other claw in the decomposition because every vertex in G has degree 3 and a graph decomposition must be edge disjoint. Therefore any leaf l of c must be the leaf of two other claws. This is true for all claws in the decomposition, so the bipartite graph can be formed by putting all the leaf nodes in one set and all the root nodes in the other set.

   Proof of $\Longleftarrow$ (if G is bipartite then it decomposes into claws): Construct the decomposition by taking each vertex in one set of the bipartite graph as roots of the claws (call this set the root set). Since each vertex in G has degree 3 and there are no edges within the root set, each vertex of the root set will have three incident edges leading to the other set (call it the leaf set), forming a claw. Since all roots are in one set of the bipartite graph, the claws must be edge disjoint. And since each vertex has degree 3, all edges crossing from the root set to the leaf set (and hence all edges in G) are accounted for. ∎

3. **Problem 1.1.47** (page 18) *Edge-transitive versus vertex-transitive.*

   (a) Let G be obtained from K_n with $n \geq 4$ by replacing each K_n with a path of two edges through a new vertex of degree 2. Prove that G is edge-transitive but not vertex-transitive.

   Let the vertices from G that were in K_n be called "original" vertices and the vertices that are added via the construction procedure above be called "new" vertices. G is not vertex-transitive because any permutation that maps an original vertex onto a new vertex or vice versa is not an isomorphism (the original vertices are all of degree $n - 1$ while the new ones are all of degree 2). On the other hand, G is edge-transitive because the end points of every edge in G consist of one new vertex and one original vertex. Since there is an automorphism for every pair of original vertices and an automorphism for every pair of new vertices, there is an automorphism for every edge in G.

   (b) Suppose that G is edge-transitive but not vertex-transitive and has no vertices of degree 0. Prove that G is bipartite.

   If G is edge-transitive but not vertex-transitive then there must be two disjoint sets of vertices such that no automorphism can map a node from one set onto a node from the other set, but such that G is vertex-transitive for pairs of vertices *within* each set. These two conditions are precisely what make G a bipartite graph.

   (c) Prove that the graph in Exercise 1.1.6 is vertex-transitive but not edge-transitive.

   All the vertices in G are of degree 3 and are part of a 3-cycle and two 4-cycles, so there is an automorphism that permutes u onto v for every $u, v \in V(G)$. However, three of the edges are in two 4-cycles, while the other six of them are in one 3-cycle and one 4-cycle, so the graph is clearly not edge-transitive.

4. **Problem 1.2.20** (page 32) Let v be a cut-vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

   When v is removed to form the $k \geq 2$ components of $G - v$, by definition of cut vertex, there are no edges between any pair of these k components. Let these components be called $C_1, C_2, ..., C_k$. Since there are *no* edges between any given pair of components, then in $\overline{G} - v$, for every $i, j \leq k$, $i \neq j$, *every* edge between the vertices in C_i and C_j is present. So $\overline{G} - v$ is connected.

5. **Problem 1.2.43** (page 34) Use induction on k to prove that every connected simple graph with an even number of edges decomposes into paths of length 2. Does the conclusion remain true if the hypothesis of connectedness is omitted?

   We prove that every connected simple graph with an even number of edges decomposes into paths of length 2 by induction on k, where k represents the number of nodes in the graph. The base case is $k = 3$ since there is no simple graph with an even number of edges with $k < 3$. The only way to form a simple graph with an even number of edges when $k = 3$ is to draw two edges which will meet at one of the vertices, forming a path of length 2. We inductively assume that the statement is true for all n, $3 \leq n \leq k$. Now consider a graph with $k + 1$ nodes. Decompose it into paths of length two as follows. Pick a node u from this graph that is of odd degree if possible. If all the nodes have even degree, then there must be a cycle in the graph, so pick u to be any node in this cycle.

Case 1. If the degree of u is odd, then remove incident edges in pairs (these are some of the 2-paths being decomposed), such that the last remaining edge from u keeps u connected to any other nodes that remain. Let v be the node connected to u. Since v is connected to the rest of the graph, there is another edge leaving v that does not go to u. Pair that edge up with the last remaining edge leaving u and remove them. Upon removal of these two edges, u has been deleted, and v is either still connected or deleted. In either case, the remainder of the graph can be decomposed by the inductive hypothesis because the remaining graph is connected and still has an even number of edges left.

Case 2. If the degree of u is even, then u was part of a cycle by our definition of u. We can pair up all the edges incident on u and remove them because each pair forms a path of length two for our decomposition. After removing u, since it was part of a cycle, we are left with a graph that is still connected of size at most k. We can apply the inductive hypothesis to decompose what's left since we removed an even number of edges, so there is an even number of edges remaining. ■

If there is no connectedness in the graph, then the conclusion is no longer true. (The proof above depends on connectedness.)

6. **Problem 1.3.18** (page 49) For $k \geq 2$, prove that a k-regular bipartite graph has no cut edge.

   Assume by way of contradiction that there is a cut edge e in a k-regular bipartite graph G. That means removing e would separate a component of G into two bipartite components, C_1 and C_2, each now with one vertex of degree $k-1$. Since all the other vertices in C_1 (respectively C_2) have degree k, there is one more edge incident on one side of C_1 (respectively C_2) than the other. But that is impossible because C_1 (respectively C_2) is bipartite.

7. **Problem 1.3.41** (page 51) Prove or disprove: If G is an n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

   This statement is true. Assume by way of contradiction that G is disconnected. That means there are at least two components of G that are not connected, call them C_1 and C_2. Assume without loss of generality that C_1 has a max-degree vertex. Since G is simple and has max degree $\lceil n/2 \rceil$, there is at least one vertex in C_1 that is connected to $\lceil n/2 \rceil$ other nodes. Therefore C_1 must be of size at least $\lceil n/2 \rceil + 1$, which means C_2 is of size at most $\lfloor n/2 \rfloor - 1$. However, every vertex in C_2 has degree at least $\lfloor n/2 \rfloor - 1$, which means each vertex is connected to at least $\lfloor n/2 \rfloor - 1$ other nodes. Thus C_2 must have at least $\lfloor n/2 \rfloor$ nodes, but this contradicts the fact that there are at most $\lfloor n/2 \rfloor - 1$ vertices in C_2.

8. True or False?

   (a) Let $e = xy$ be a cut edge in a connected graph G with $|V(G)| > 2$. Then x or y is a cut vertex in G.

   True. Since $e = xy$ is a cut edge, that means removing it leaves x and y in two separate components. Since $|V| > 2$ and G is connected, at least one of these components has more than one vertex. Without loss of generality assume the connected component containing more than one vertex in $G - e$ is the one containing x. Since removing x from G removes e, and leaves behind two components: one with y and the other with the vertices that were connected to x, then x is a cut vertex of G. Note that y may not be a cut vertex because it may not have any other edges incident on it besides e.

(b) A simple graph can not have exactly 3 automorphisms.

True. A simple graph can have one automorphism (the identity), and it can have two automorphisms (the identity and one permutation of the vertices), but no graph has only the identity and two other permutations in its automorphism group.

(c) Let G be a bipartite graph with bipartition (V_1, V_2). Then either V_1 or V_2 is a largest independent set in G.

False. Consider a bipartite graph with three nodes (v_{11}, v_{12}, v_{13}) in V_1 and three nodes (v_{21}, v_{22}, v_{23}) in V_2. Place an edge from v_{11} to v_{21} and place no other edges. Thus $|V_1| = |V_2| = 3$ but the largest independent set has five elements: $\{v_{11}, v_{12}, v_{13}, v_{22}, v_{23}\}$

(d) A simple bipartite graph on 11 vertices can have 31 edges.

False. To maximize the number of edges in a bipartite graph on 11 vertices, you need to put 5 vertices in one set and 6 in the other, then draw in the edges for $K_{5,6}$, which total 30.

(e) The sequence $(5, 5, 4, 4, 3, 3, 2)$ is a graphic sequence.

True. Using Havel/Hakimi's technique you can reduce the problem to being a graphic sequence if and only if $(1, 1, 0)$ is a graphic sequence, and $(1, 1, 0)$ is a graphic sequence because you can construct a graph of two components: one vertex of degree zero, and an edge, each end point of which is a vertex of degree one. The reduction goes $(5, 5, 4, 4, 3, 3, 2) \rightarrow (4, 3, 3, 2, 2, 2) \rightarrow (2, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1) \rightarrow (1, 1, 0)$.