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CS 3150 Randomness and Computation
Homework 1 **Due:** Monday, Jan 9, 2005

Chapter 1, Exercise 1.22 (page 18)

(a) Consider the set $\{1, ..., n\}$. We generate a subset X of this set as follows: a fair coin is flipped independently for each element of the set; if the coin lands heads then the element is added to X, and otherwise it is not. Argue that the resulting set X is equally likely to be any one of the 2^n possible subsets.

Each of the n elements can either be added to X or not (depending on each of the n coin flips), accounting for the total of 2^n possible subsets. For any given subset S of $\{1, ..., n\}$, let Π_S be the string of n ones and zeroes where $\Pi_S(i) = 1$ if $i \in S$ and $\Pi_S(i) = 0$ otherwise. The probability that the ith coin flip lands heads when $\Pi_S(i) = 1$ is 1/2 (and likewise for landing tails when $\Pi_S(i) = 0$). So the set S is chosen with probability $(\frac{1}{2})^n = \frac{1}{2^n}$. Since any given subset S is chosen with an equal probability of $\frac{1}{2^n}$, X is equally likely to be any one of the 2^n possible subsets. (Also see Lemma 1.5 on page 8 of text.)

(b) Suppose that two sets X and Y are chosen independently and uniformly at random from all the 2^n subsets of $\{1, ..., n\}$. Determine $\mathbf{Pr}[X \subseteq Y]$ and $\mathbf{Pr}[X \cup Y = \{1, ..., n\}]$.

By the law of total probability (theorem 1.6 on page 9), we know that

$$\mathbf{Pr}[X \subseteq Y] = \sum_{k=0}^{n} \mathbf{Pr}[X \subseteq Y \mid |Y| = k] \cdot \mathbf{Pr}[|Y| = k]. \tag{1}$$

We will proceed to solve the right side of (1). If $|Y| = k$, there are 2^k subsets of Y. Call these subsets $S_1, S_2, ..., S_{2^k}$. By part (a), X is equally likely to be any of these subsets and $\mathbf{Pr}[X = S_i] = \frac{1}{2^n}$ for $1 \leq i \leq 2^k$. Therefore

$$\begin{aligned}
\mathbf{Pr}[X \subseteq Y \mid |Y| = k] &= \mathbf{Pr}\left[\cup_{i=1}^{2^k} X = S_i\right] \\
&= \mathbf{Pr}[X = S_1] + \mathbf{Pr}[X = S_2] + ... + \mathbf{Pr}[X = S_{2^k}] \\
&= \sum_{i=1}^{2^k} \frac{1}{2^n} = \frac{2^k}{2^n} = 2^{k-n}.
\end{aligned} \tag{2}$$

Also,

$$\begin{aligned}
\mathbf{Pr}[|Y| = k] &= \frac{\text{number of subsets of } \{1, ..., n\} \text{ of size } k}{\text{total number of subsets of } \{1, ...n\}} \\
&= \frac{\binom{n}{k}}{2^n} = \binom{n}{k} 2^{-n}.
\end{aligned} \tag{3}$$

Plugging (2) and (3) into equation (1) gives us:

$$\begin{aligned}
\mathbf{Pr}[X \subseteq Y] &= \sum_{k=0}^{n} 2^{k-n}\binom{n}{k}2^{-n} \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^{n-k} = \frac{1}{2^n}\left(\frac{1}{2}+1\right)^n \qquad (4)\\
&= \frac{1}{2^n}\left(\frac{3}{2}\right)^n = \left(\frac{3}{4}\right)^n, \qquad (5)
\end{aligned}$$

where (4) is due to the Binomial Theorem, which states that $\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k = (a+b)^n$.

This result can be directly applied to the second part of the problem as follows.

$$\begin{aligned}
\mathbf{Pr}[X \cup Y = \{1, ..., n\}] &= \mathbf{Pr}[\{1, ..., n\} - X \subseteq Y] \qquad (6)\\
&= \mathbf{Pr}[X \subseteq Y]. \qquad (7)
\end{aligned}$$

The reasoning behind (6) is that after removing the set X from the set {1,...,n}, if all elements that remain are in the set Y, then $X \cup Y = \{1, ..., n\}$. The second line (7) is due to what we learned in (5): if A and B are two sets chosen independently and uniformly at random from the subsets of $\{1, ..., n\}$, then $\mathbf{Pr}[A \subseteq B] = \left(\frac{3}{4}\right)^n$. Part (a) says the set A has as much of a chance of being equal to a given set X as it does $\{1, ..., n\} - X$, so $\mathbf{Pr}[X \cup Y = \{1, ..., n\}] = \left(\frac{3}{4}\right)^n$.