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Sherif Khattab, and Jonathan Misurda ("Group A") **Homework 11**
CS 3150 Randomness and Computation **Due:** Wed, March 15, 2006

Chapter 10, Exercise 10.10, page 269

A Δ-coloring C of an undirected graph $G = (V, E)$ is an assignment labeling each vertex with a number, representing a color, from the set $\{1, 2, ..., \Delta\}$. An edge (u, v) is *improper* if both u and v are assigned the same color. Let $I(C)$ be the number of improper edges of a coloring C. Design a Markov chain based on the Metropolis algorithm such that, in the stationary distribution, the probability of a coloring is proportional to $\lambda^{I(C)}$ for a given constant $\lambda > 0$. Pairs of states of the chain are connected if they correspond to pairs of colorings that differ in just one vertex.

The state space is all of the possible Δ-colorings of G. The neighborhood of a state x, which is a coloring of G, is all the colorings that differ from x in exactly one vertex, and is denoted $N(x)$. This state space is irreducible under the Markov chain since its graph representation H will be strongly connected (i.e. any state in H is reachable from any other state in H).

We design a Markov chain, whose states are colorings of the graph G, as follows:

1. Start with X_0 as an arbitrary coloring of G. (Let $i = 0$.)
2. To compute X_{i+1}:
   (a) choose a vertex v uniformly at random from V;
   (b) define $X_i(v)$ to be the color of v in X_i;
   (c) choose a color c uniformly at random from $\{1, 2, ..., \Delta\}$;
   (d) if $X_i(v) \neq c$ and coloring v with c decreases $I(X_i)$ by k, then let X_{i+1} be X_i with v colored c with probability $\min(\frac{1}{\lambda^k}, 1)$;
   (e) if $X_i(v) \neq c$ and coloring v with c increases $I(X_i)$ by k, then let X_{i+1} be X_i with v colored c with probability $\min(1, \lambda^k)$;
   (f) if $X_i(v) \neq c$ and coloring v with c neither increases nor decreases $I(X_i)$, then let X_{i+1} be X_i with v colored c with probability 1;
   (g) otherwise, let $X_{i+1} = X_i$.

To show the probability of being in a coloring C is proportional to $\lambda^{I(C)}$ for the above Markov chain, we have to show that $\pi_x = \lambda^{I(x)}/B$ is the stationary distribution of this Markov chain, where $I(x)$ is the number of improper edges in coloring x, and $B = \sum_x \lambda^{I(x)}$. The algorithm (in steps a and c) first picks a node-color pair at random from the $M = (|V|)(\Delta)$ possibilities to be the move under consideration. (Note that $M \geq |N(x)| = (|V|)(\Delta - 1)$.) If x is the current state and y is the state that we might move to by coloring the randomly-picked node with the randomly-picked color, note that π_y/π_x is λ^k if the number of improper edges increases by k in the move, it is $1/\lambda^k$ if the number of improper edges decreases by k in the move, and it is 1 if the number of improper edges doesn't change. So the Markov chain picks a specific y with probability $1/M$, then moves from x to y with probabilty $\min(1, \pi_y/\pi_x)$. Also note that if $x \neq y$, then y is necessarily a neighbor of x since any coloring that differs in just one vertex is a neighbor. Therefore we have the probability that we will go from state x to state y in one step:

$$P_{x,y} = \begin{cases} (1/M)\min(1, \pi_y/\pi_x) & \text{if } x \neq y \text{ and } y \in N(x) \\ 0 & \text{if } x \neq y \text{ and } y \notin N(x) \\ 1 - \sum_{y \neq x} P_{x,y} & \text{if } x = y. \end{cases}$$

Now, by Lemma 10.8 on page 265, we have shown the stationary distribution is given by the probabilities π_x.